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Directional singularities of the 'rod solution' in Einstein–Maxwell–Yukawa fields

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Abstract. The generalized 'rod solution' of Einstein–Maxwell–Yukawa fields is obtained and the directional behaviour of the singularities is discussed. The solution obtained exhibits directional singularities for combined zero-mass scalar meson and electromagnetic fields under certain conditions on the parameters. It has been observed that the presence of a source-free electromagnetic field does not affect the directional behaviour of the singularities when only a zero-mass meson field is present. This has been investigated by Gautreau.

1. Introduction

The study of singular behaviour of the coupled gravitational and zero rest-mass scalar meson field has been investigated by Gautreau (1969). He has examined the static, cylindrically symmetric fields corresponding to the Curzon solution (Curzon 1924) and those due to the Newtonian potential of a rod in the presence of zero rest-mass scalar meson field to determine the changes that occur in their singular structure. In our earlier paper (Datta and Rao 1973), we have extended this study corresponding to the Curzon solution, where the space–time, in addition to zero rest-mass scalar meson field, contains a source-free electromagnetic field. It may be noted that, whereas Gautreau's solution (1969) and the vacuum solution (Gautreau and Anderson 1967), contains directional singularities, our solution (Datta and Rao 1973) has not revealed such phenomena. In the present paper, we have taken the solution corresponding to the Newtonian potential of a rod in the presence of source-free electromagnetic and zero rest-mass scalar meson fields and studied the directional behaviour of the singularities that occur. Three possible cases arise in our analysis according to the restrictions on the parameters. Directional singularities ensue in two of the cases and in the third case no such behaviour is present. It may be remarked that this third possibility leads to the solution discussed earlier by the authors (Datta and Rao 1973) which may be looked upon as a special case when the 'rod' collapses to a 'point'.

In § 2, we have obtained the solution corresponding to the Newtonian potential of a rod in the presence of combined zero rest-mass scalar meson and electromagnetic fields. Section 3 deals with the study of the Kretschmann curvature invariant and it is shown that the presence of zero rest-mass scalar meson and electromagnetic fields introduce directional behaviour in the singularities under certain conditions on the parameters. Calculations of the area of the equipotential surfaces and the proper lengths of the closed curves are presented in § 4. Finally, we have summarized our analysis in § 5.

2. Solution of the coupled electromagnetic and zero rest-mass scalar meson fields

In this section, we have obtained the solutions of the coupled electromagnetic and zero rest-mass scalar meson fields for the Newtonian potential of a rod.

We consider the cylindrically symmetric static metric

$$ds^2 = -\exp(2\nu - 2\lambda)(dr^2 + dz^2) - r^2 e^{-2\lambda} d\phi^2 + e^{2\lambda} dt^2 \tag{1}$$

where λ, ν are functions of the coordinates r and z only.

The Newtonian potential of a rod of length $2a$ and mass per unit length $\epsilon/2$ is given by

$$\begin{aligned} \lambda &= \frac{\epsilon}{2} \lg \left(\frac{l_1 + l_2 - 2a}{l_2 + l_1 + 2a} \right) \\ \nu &= \frac{\epsilon^2}{2} \lg \left(\frac{(l_1 + l_2)^2 - 4a^2}{4 l_1 l_2} \right) \end{aligned} \tag{2}$$

where $l_2^2 = r^2 + (z+a)^2$ and $l_1^2 = r^2 + (z-a)^2$.

Following the technique developed by Janis *et al* (1969), the solution of the field equations for the coupled electromagnetic and zero rest-mass scalar meson field is obtained as

$$V = AA' \frac{\epsilon}{2} \lg \left(\frac{l_2 + l_1 - 2a}{l_2 + l_1 + 2a} \right)$$

where $A'(=[1 + (\kappa A^2/8\pi)]^{1/2})$ is a constant

$$F_{14} = -\left(\frac{8\pi}{\kappa} \right)^{1/2} A' \frac{\epsilon}{2} \operatorname{cosech}^2 \left(A' \frac{\epsilon}{2} \lg \frac{l_2 + l_1 - 2a}{l_2 + l_1 + 2a} \right) \frac{(l_2 + l_1) 4ar}{l_2 l_1 (l_2 + l_1 + 2a)(l_2 + l_1 - 2a)}$$

$$F_{24} = -\left(\frac{8\pi}{\kappa} \right)^{1/2} A' \frac{\epsilon}{2} \operatorname{cosech}^2 \left(A' \frac{\epsilon}{2} \lg \frac{l_2 + l_1 - 2a}{l_2 + l_1 + 2a} \right) \frac{(l_2 + l_1) 4az - 4a^2 (l_2 - l_1)}{l_2 l_1 (l_2 + l_1 + 2a)(l_2 + l_1 - 2a)}$$

$$\lambda = \lg \operatorname{cosech} \left(A' \frac{\epsilon}{2} \lg \frac{l_2 + l_1 - 2a}{l_2 + l_1 + 2a} \right)$$

$$\nu = \frac{\epsilon^2}{2} \lg \left(\frac{(l_2 + l_1)^2 - 4a^2}{4 l_2 l_1} \right) \tag{3}$$

where F_{ij} is the electromagnetic field tensor and V is the scalar potential.

3. Study of the Kretschmann curvature invariant

The Kretschmann curvature invariant α for the solution (3) is

$$\alpha \equiv R_{hijk}R^{hijk} = \exp(-4\nu+4\lambda)\Big[(\nu_{,22}+\nu_{,11}-\lambda_{,22}-\lambda_{,11})^2+\Big(\lambda_{,11}-\lambda_{,2}^2 -\lambda_{,1}\nu_{,1}+\lambda_{,2}\nu_{,2}+\frac{1}{r}(\lambda_{,1}+\nu_{,1})\Big)^2+(-\lambda_{,11}-2\lambda_{,1}^2+\lambda_{,1}\nu_{,1}-\lambda_{,2}\nu_{,2}+\lambda_{,2}^2)^2 + \Big(\lambda_{,22}+\lambda_{,1}\nu_{,1}-\lambda_{,2}\nu_{,2}-\lambda_{,1}^2+\frac{1}{r}(\lambda_{,1}-\nu_{,1})\Big)^2+(-\lambda_{,22}+\lambda_{,1}^2-2\lambda_{,2}^2 -\lambda_{,1}\nu_{,1}+\lambda_{,2}\nu_{,2})^2+\Big(\lambda_{,12}+\lambda_{,1}\lambda_{,2}-\lambda_{,1}\nu_{,2}-\lambda_{,2}\nu_{,1}+\frac{\nu_{,2}}{r}\Big)^2 +(-\lambda_{,12}-3\lambda_{,1}\lambda_{,2}+\lambda_{,1}\nu_{,2}+\lambda_{,2}\nu_{,1})^2+\Big(\lambda_{,1}^2+\lambda_{,2}^2-\frac{\lambda_{,1}}{r}\Big)^2\Big]. \tag{4}$$

In the above expression the value of the last term is given as

$$\exp(-4\nu+4\lambda)\Big(\lambda_{,1}^2+\lambda_{,2}^2-\frac{\lambda_{,1}}{r}\Big)^2 = \frac{4A'^2\epsilon^2a^2\coth^2\Big(A'\frac{\epsilon}{2}\lg\frac{l_2+l_1-2a}{l_2+l_1+2a}\Big)}{l_2^2l_1^2((l_2+l_1)^2-4a^2)^2}\Big[4A'^2\epsilon^2a^2\coth^2\Big(A'\frac{\epsilon}{2}\lg\frac{l_2+l_1-2a}{l_2+l_1+2a}\Big) +4A'\epsilon a\coth\Big(A'\frac{\epsilon}{2}\lg\frac{l_2+l_1-2a}{l_2+l_1+2a}\Big)(l_2+l_1)-(l_2+l_1)^2\Big] \tag{5}$$

which has the following limiting behaviour: as $r\to 0$, $z\to \pm a$, α tends to infinity, a finite value or zero according to whether $A'\epsilon$ is less than, equal to, or greater than two. It has been verified that the limiting behaviour is same for all the other terms.

4. Area of equipotential surfaces and proper lengths

Using Schwarzschild-like coordinates related to Weyl coordinates by

$$R=\tfrac{1}{2}(l_2+l_1)+a,\qquad \cos\theta=\frac{1}{2a}(l_2-l_1),$$
$$z=(R-a)\cos\theta,\qquad r=(R^2-2aR)^{1/2}\sin\theta \tag{6}$$

the metric (1) takes the form

$$ds^2=-\frac{(R)^{\epsilon^2-2}(R-2a)^{\epsilon^2}[dR^2/(1-2a/R)+R^2\,d\theta^2]}{(R^2-2aR+a^2\sin^2\theta)^{\epsilon^2-1}\operatorname{cosech}^2\Big(A'\frac{\epsilon}{2}\lg\frac{l_2+l_1-2a}{l_2+l_1+2a}\Big)} -\frac{R(R-2a)\sin^2\theta}{\operatorname{cosech}^2\Big(A'\frac{\epsilon}{2}\lg\frac{l_2+l_1-2a}{l_2+l_1+2a}\Big)}d\phi^2+\operatorname{cosech}^2\Big(A'\frac{\epsilon}{2}\lg\frac{l_2+l_1-2a}{l_2+l_1+2a}\Big)dt^2. \tag{7}$$

The area of equipotential surfaces g_{44} = constant, t = constant for the metric (7) is given by

$$\mathscr{A} = \int_0^{2\pi}\int_0^{\pi}(g_{\theta\theta}g_{\phi\phi})^{1/2}\,\mathrm{d}\theta\,\mathrm{d}\phi = \frac{\pi}{2}(R)^{\epsilon^2/2+\frac{1}{2}+A'\epsilon}(R-2a)^{\epsilon^2/2+\frac{1}{2}-A'\epsilon}\left[\left(\frac{R-2a}{R}\right)^{A'\epsilon}-1\right]^2 \times\int_{-1}^{+1}\frac{\mathrm{d}\mu}{[(R-a)^2-a^2\mu^2]^{\epsilon^2/2-\frac{1}{2}}}. \tag{8}$$

For $\epsilon^2 \neq 3$ or 5, $\mathscr{A}$ can be written as

$$\mathscr{A} = \frac{\pi\left[\left(\frac{R-2a}{R}\right)^{A'\epsilon}-1\right]^2}{(R-a)^2(\epsilon^2-3)}\left\{(R)^{2+A'\epsilon}(R-2a)^{2-A'\epsilon}+\frac{(\epsilon^2-4)}{(R-a)^2(\epsilon^2-5)}\left[(R)^{3+A'\epsilon}(R-2a)^{3-A'\epsilon}\right.\right.$$
$$\left.\left.+\left(\frac{\epsilon^2}{2}-3\right)(R)^{\epsilon^2/2+\frac{1}{2}+A'\epsilon}(R-2a)^{\epsilon^2/2+\frac{1}{2}-A'\epsilon}\int_{-1}^{+1}\frac{\mathrm{d}\mu}{[(R-a)^2-a^2\mu^2]^{\epsilon^2/2-\frac{1}{2}}}\right]\right\} \tag{9}$$

while for $\epsilon^2 = 3$:

$$\mathscr{A} = \frac{\pi[(R-2a/R)^{A'\epsilon}-1]^2}{2a(R-a)}(R)^{2+A'\epsilon}(R-2a)^{2-A'\epsilon}\lg\frac{R}{R-2a} \tag{10}$$

and for $\epsilon^2 = 5$:

$$\mathscr{A} = \frac{\pi\left[\left(\frac{R-2a}{R}\right)^{A'\epsilon}-1\right]^2}{2(R-a)^2}\left((R)^{2+A'\epsilon}(R-2a)^{2-A'\epsilon}+\frac{(R)^{3+A'\epsilon}(R-2a)^{3-A'\epsilon}}{2a(R-a)}\lg\frac{R}{R-2a}\right). \tag{11}$$

From these expressions for $\mathscr{A}$ we observe that as $R \to 2a$, $\mathscr{A} \to 0$ for $\epsilon < 2$ (and therefore $A'\epsilon < 2$ since $A' < 1$); $\mathscr{A} \to 0$ for $\epsilon = 2$ (ie $A'\epsilon < 2$) and $\mathscr{A}$ tends to zero, a finite value or infinity for $\epsilon > 2$ according to whether $A'\epsilon$ is less than, equal to, or greater than two.

Furthermore, the proper length L_ϕ of the closed azimuthal curve $\theta = \pi/2$, is obtained as

$$L_\phi \equiv \int_0^{2\pi}(-g_{\phi\phi})^{1/2}\,\mathrm{d}\phi = \pi(R)^{\frac{1}{2}+A'\epsilon/2}(R-2a)^{\frac{1}{2}-A'\epsilon/2}\left[\left(\frac{R-2a}{R}\right)^{A'\epsilon}-1\right]. \tag{12}$$

The proper length L_θ of a polar curve, ϕ = constant, has the value

$$L_\theta = 2\int_0^{\pi}(-g_{\theta\theta})^{1/2}\,\mathrm{d}\theta = (R)^{\epsilon^2/2+A'\epsilon/2}(R-2a)^{\epsilon^2/2-A'\epsilon/2}\left[\left(\frac{R-2a}{R}\right)^{A'\epsilon}-1\right] \times\int_0^{\pi}\frac{\mathrm{d}\theta}{[R^2-2aR+a^2\sin^2\theta]^{\epsilon^2/2-\frac{1}{2}}}. \tag{13}$$

5. Conclusions

We can summarize in short the directional behaviour or otherwise of the solution in table 1.

Table 1.

	Kretschmann curvature invariant, α	Area of equipotential surfaces $\mathscr{A}$	Structure of singularity
$A'\epsilon < 2$	∞	0	No directional behaviour. The space t = constant may be regarded as simply connected.
$A'\epsilon = 2$	a finite value	a finite value	Directional singularity.
$A'\epsilon > 2$	0	∞	Directional behaviour. The space t = constant may be regarded as multiply connected.

Following arguments similar to those employed by Gautreau (1969) in dealing with the problem in the absence of electromagnetic field, it is seen from (8), (12) and (13) that in the case corresponding to $\epsilon = 1 (A'\epsilon < 1)$ only, our solution exhibits a point singularity, as both the area of equipotential surfaces and the proper lengths of the closed curves on these surfaces tend to zero as $R \rightarrow 2a$. It may also be mentioned that the addition of electromagnetic field does not qualitatively alter the topological structure of the singularities as when only scalar field is present. It may further be noted that our earlier work (Datta and Rao 1973), which may be considered as a special case of the 'rod solution' when the rod shrinks to a 'point', is obtainable under the condition $A'\epsilon < 1$ as it entails the topology of a point singularity (Winicour *et al* 1968).

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